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The Surreal Numbers and Combinatorial Games

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Appendix: Definitions

Axiom 1: For any two *sets of numbers* L and R,

$$\exists \text{ the number } \{L|R\} \iff \nexists x \in L : x \geqslant y, \forall y \in R.$$

Axiom 2: For any two numbers $x = \{X^L|X^R\}$ and $y = \{Y^L|Y^R\}$,

$$x \leqslant y \iff \nexists x^L \in X^L : x^L \geqslant y \wedge \nexists y^R \in Y^R : y^R \leqslant x.$$

$x = y \iff x \leqslant y \wedge y \leqslant x.$

$x < y \iff x \leqslant y \wedge y \nleqslant x.$

$x \equiv y$: The two numbers x and y are identical if and only if $X^L = Y^L$ and $X^R = Y^R$.

Birthday: The birthday of a number is day on which it is first constructed.

Simplicity: We say that a number x is simpler than another number y if x has an earlier birthday than y.

Natural Form: A number x is in its natural form if it has at most one left option, and at most one right option, and all its options are strictly simpler than x.

Arithmetic on the Surreal Numbers

$$\begin{aligned} x + y &= \{X^L + y, x + Y^L | X^R + y, x + Y^R\} \\ &= \{x_1^L + y, x_2^L + y, ..., x + y_1^L, x + y_2^L, ... | x_1^R + y, x_2^R + y, ..., x + y_1^R, x + y_2^R, ...\} \end{aligned}$$

$$\begin{aligned} -x &= \{-X^R | -X^L\} \\ &= \{-x_1^R, -x_2^R, ... | -x_1^L, -x_2^L, ...\} \end{aligned}$$

$$xy = \{X^L y + xY^L - X^L Y^L, X^R y + xY^R - X^R Y^R | X^L y + xY^R - X^L Y^R, X^R y + xY^L - X^R Y^L\}$$
$$\text{where for sets } A \text{ and } B,\ Ax = \{ax : a \in A\}, \text{ and } AB = \{ab : a \in A, b \in B\}$$

$$\frac{1}{x} = \left\{0, \frac{1 + (x^R - x)\frac{1}{x}^L}{x^R}, \frac{1 + (x^L - x)\frac{1}{x}^R}{x^L} \middle| \frac{1 + (x^L - x)\frac{1}{x}^L}{x^L}, \frac{1 + (x^R - x)\frac{1}{x}^R}{x^R}\right\}$$
$$\text{where } (1/x)^L \text{ and } (1/x)^R \text{ are the already computed elements of } 1/x$$