

2019

The Surreal Numbers and Combinatorial Games

Holden, D.

<http://hdl.handle.net/10026.1/14684>

The Plymouth Student Scientist

Holden, D. (2019) 'The Surreal Numbers and Combinatorial Games', The Plymouth Student Scientist, 12(1), p. 63-134.

All content in PEARL is protected by copyright law. Author manuscripts are made available in accordance with publisher policies. Please cite only the published version using the details provided on the item record or document. In the absence of an open licence (e.g. Creative Commons), permissions for further reuse of content should be sought from the publisher or author.

Appendix: Definitions

Axiom 1: For any two sets of numbers L and R ,

$$\exists \text{ the number } \{L|R\} \iff \nexists x \in L : x \geq y, \forall y \in R.$$

Axiom 2: For any two numbers $x = \{X^L|X^R\}$ and $y = \{Y^L|Y^R\}$,

$$x \leq y \iff \nexists x^L \in X^L : x^L \geq y \wedge \nexists y^R \in Y^R : y^R \leq x.$$

$$x = y \iff x \leq y \wedge y \leq x.$$

$$x < y \iff x \leq y \wedge y \not\leq x.$$

$x \equiv y$: The two numbers x and y are identical if and only if $X^L = Y^L$ and $X^R = Y^R$.

Birthday: The birthday of a number is day on which it is first constructed.

Simplicity: We say that a number x is simpler than another number y if x has an earlier birthday than y .

Natural Form: A number x is in its natural form if it has at most one left option, and at most one right option, and all its options are strictly simpler than x .

Arithmetic on the Surreal Numbers

$$\begin{aligned} x + y &= \{X^L + y, x + Y^L | X^R + y, x + Y^R\} \\ &= \{x_1^L + y, x_2^L + y, \dots, x + y_1^L, x + y_2^L, \dots | x_1^R + y, x_2^R + y, \dots, x + y_1^R, x + y_2^R, \dots\} \end{aligned}$$

$$\begin{aligned} -x &= \{-X^R | -X^L\} \\ &= \{-x_1^R, -x_2^R, \dots | -x_1^L, -x_2^L, \dots\} \end{aligned}$$

$$\begin{aligned} xy &= \{X^L y + x Y^L - X^L Y^L, X^R y + x Y^R - X^R Y^R | X^L y + x Y^R - X^L Y^R, X^R y + x Y^L - X^R Y^L\} \\ &\text{where for sets } A \text{ and } B, Ax = \{ax : a \in A\}, \text{ and } AB = \{ab : a \in A, b \in B\} \end{aligned}$$

$$\frac{1}{x} = \left\{ 0, \frac{1 + (x^R - x)\frac{1}{x}^L}{x^R}, \frac{1 + (x^L - x)\frac{1}{x}^R}{x^L} \mid \frac{1 + (x^L - x)\frac{1}{x}^L}{x^L}, \frac{1 + (x^R - x)\frac{1}{x}^R}{x^R} \right\}$$

where $(1/x)^L$ and $(1/x)^R$ are the already computed elements of $1/x$